

EXTENSIONS OF SIMPLE MODULES AND THE CONVERSE OF SCHUR'S LEMMA

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ABSTRACT. The *converse of Schur's lemma* (or *CSL*) condition on a module category has been the subject of considerable study in recent years. In this note we extend that work by developing basic properties of module categories in which the CSL condition governs modules of finite length.

1. INTRODUCTION

Schur's Lemma states that for any ring R and any simple module M_R , the endomorphism ring $\text{End}(M_R)$ is a division ring. In this note we are interested in the converse of Schur's Lemma (CSL), i.e. whether for a given module category \mathcal{C} , every object in \mathcal{C} whose endomorphism ring is a division ring is in fact simple. If this is the case, we say that \mathcal{C} *has CSL*. The case that has received almost exclusive attention in the literature (e.g. see [1], [2], [7], [10], [9], [14]) is $\mathcal{C} = \text{Mod}_R$, the category of right R -modules. Here we will focus on the case $\mathcal{C} = \mathcal{FL}_R$, the category of right R -modules of finite length.

We propose to separate the study of rings R which satisfy CSL for \mathcal{FL}_R from the study of rings which satisfy CSL for \mathcal{FL}_R but not for Mod_R , since the two properties relate to different topics: extensions of simples versus constructions of large modules.

It turns out that the CSL property for finite length modules—and sometimes the CSL property for all modules—is controlled by the following combinatorial information:

Definition 1.1. Let R be a ring. Recall that the *right Gabriel quiver* (or *right Ext-quiver*) of R is the directed graph Q consisting of the following data:

- The points of Q are in bijective correspondence with the isomorphism classes of simple right R -modules.
- There is an arrow $i \rightarrow j$ in Q whenever the corresponding simple modules S_i and S_j extend, i.e. $\text{Ext}_R^1(S_i, S_j) \neq \{0\}$.

We say that the right Gabriel quiver is *totally disconnected* if there are no arrows between any two different points.

For example, the right Gabriel quiver of a semisimple ring is a disjoint union of finitely many points. The right Gabriel quiver of the discrete valuation ring $R = k[x]_{(x)}$ (for k a field) has one point and one loop.

The reader should be aware that the literature contains some variants of the definition we give here. In the classical setting where R is a finite-dimensional

Date: May 13, 2009.

1991 Mathematics Subject Classification. Primary 16D90, 16G20, 16S50.

Key words and phrases. Converse of Schur's Lemma, Gabriel quiver.

algebra over a field, some authors adopt the convention that in the right Gabriel quiver of R the arrow between the vertices corresponding to S_i and S_j carries as label the pair given by the dimensions of $\text{Ext}_R^1(S_i, S_j)$ as a vector space over $\text{End}(S_j)_R$ and $\text{End}(S_i)_R$ respectively.

Theorem 1.2. *Let R be any ring. Then \mathcal{FL}_R has CSL if and only if the right Gabriel quiver of R is totally disconnected.*

We will prove Theorem 1.2 in Section 2.

In general, for \mathcal{FL}_R to have CSL is a considerably weaker condition than for \mathcal{Mod}_R to have CSL. The distinction between CSL on \mathcal{FL}_R and CSL on \mathcal{Mod}_R is illustrated in the following examples.

Example 1.3. Let R be any commutative ring whatsoever. Then \mathcal{FL}_R has CSL. To see this, suppose M is an R -module of finite length such that $\text{End}(M_R)$ is a division ring. If M were not simple, then by [14, Corollary], M_R would be isomorphic to the field of fractions of R/\mathfrak{p} where $\mathfrak{p} = \text{ann}^R(M)$ is a prime but not maximal ideal of R , contradicting the hypothesis that M has finite length.

By contrast, in [14] it is shown that for a commutative ring R , the category \mathcal{Mod}_R has CSL if and only if R has Krull dimension 0.

We infer from Theorem 1.2 that the (right) Gabriel quiver of any commutative ring is totally disconnected.

Example 1.3 suggests a further reason why the (not necessarily commutative) rings R for which \mathcal{FL}_R has CSL are an interesting object of study: they include all commutative rings, so this condition is a new sort of generalization of commutativity.

Example 1.4 (J. H. Cozzens [4]). Let K be an algebraically closed field of positive characteristic p , let $\varphi: x \mapsto x^{p^n}$ be a Frobenius automorphism on K , and let $k = K^{\langle \varphi \rangle}$ be the fixed field of φ . Then the skew Laurent polynomial ring $R = K[x, x^{-1}; \varphi]$ has, up to isomorphism, a unique simple right module S , which is injective. Thus, every finite length right R -module is semisimple, and hence \mathcal{FL}_R has CSL.

Nevertheless, \mathcal{Mod}_R does not have CSL. It is easy to show that if R is a right or left Ore domain, then \mathcal{Mod}_R has CSL if and only if R is a division ring. In the present example R is a simple noetherian domain, hence an Ore domain.

Note that the category \mathcal{FL}_R is equivalent to the category \mathcal{FL}_k , which obviously has CSL (as even \mathcal{Mod}_k does).

We can generalize this example using [13, Theorem A]. Let $\text{rad}(A)$ denote the Jacobson radical of a ring A . Recall that a finite-dimensional k -algebra A is called *elementary* if $A/\text{rad}(A)$ is a finite direct product of copies of k .

Proposition 1.5. *Let A be a finite-dimensional elementary algebra over a finite field whose right Gabriel quiver is totally disconnected. Then there exists a noetherian ring R such that the following properties hold:*

- (i) *The categories \mathcal{FL}_A and \mathcal{FL}_R are equivalent.*
- (ii) *Both \mathcal{FL}_A and \mathcal{FL}_R have CSL.*
- (iii) *The category \mathcal{Mod}_A has CSL, but the category \mathcal{Mod}_R does not.*

On the other hand, for semiprimary rings the CSL property for all modules is controlled by the right Gabriel quiver, i.e. it is controlled by the CSL property for

finite length modules. Recall that a ring R is said to be *semiprimary* if the Jacobson radical $\text{rad}(R)$ is nilpotent and $R/\text{rad}(R)$ is a semisimple ring. Semiprimary rings figure prominently in our main object of study here: it is well known that the endomorphism ring of a finite length module is semiprimary.

Theorem 1.6. *Let R be a semiprimary ring. The following conditions are equivalent:*

- (i) \mathcal{FL}_R has CSL.
- (ii) The right Gabriel quiver of R is totally disconnected.
- (iii) R is a finite direct product of full matrix rings over local rings.
- (iv) Mod_R has CSL.

We defer the proofs of Proposition 1.5 and Theorem 1.6 to Section 3. The literature contains results akin to Theorem 1.6, such as the following.

Theorem 1.7. *Let R be a one-sided noetherian ring or a perfect ring. Then Mod_R has CSL if and only if R is a finite direct product of full matrix rings over local perfect rings.*

The left noetherian case is covered by [2, Theorem 1], the right noetherian case by [5, Theorem 3.4], and the perfect case by [1, Theorem 1.2].

Example 1.8. Let k be a field of characteristic 0, and let $\mathbb{A}_1(k) = k\langle x, y \rangle / (xy - yx - 1)$ be the first Weyl algebra over k . If

$$S_1 = \mathbb{A}_1(k)/x\mathbb{A}_1(k) \quad \text{and} \quad S_2 = \mathbb{A}_1(k)/(x + y)\mathbb{A}_1(k),$$

then by [11, Proposition 5.6, Theorem 5.7], S_1 and S_2 are nonisomorphic simple right $\mathbb{A}_1(k)$ -modules for which $\text{Ext}_{\mathbb{A}_1(k)}^1(S_1, S_2) \neq \{0\}$. Therefore, the right Gabriel quiver of $\mathbb{A}_1(k)$ is not totally disconnected, so Theorem 1.2 tells us $\mathcal{FL}_{\mathbb{A}_1(k)}$ does not have CSL.

The conclusion of Example 1.8 can be extended from $\mathbb{A}_1(k)$ to certain *generalized Weyl algebras*; see [3, Theorem 1.1] for details.

Example 1.9. Let R be a right bounded Dedekind prime ring. Then \mathcal{FL}_R has CSL. To prove this, first note that if R is right primitive then by [6, Theorem 4.10] it is simple artinian (so in this case even Mod_R has CSL). Now assume R is not right primitive. Suppose S_1 and S_2 are arbitrary nonisomorphic simple right R -modules. Then $\text{ann}_R^R(S_1) = \mathfrak{m}_1$ and $\text{ann}_R^R(S_2) = \mathfrak{m}_2$ are maximal ideals of R and $\mathfrak{m}_1 \neq \mathfrak{m}_2$, by [12, Theorem 3.5]. By [6, Theorem 1.2, Proposition 2.2], \mathfrak{m}_1 and \mathfrak{m}_2 are invertible ideals. We can therefore apply [8, Proposition 1] to conclude that $\text{Ext}_{\mathbb{A}_1(k)}^1(S_1, S_2) = \{0\}$. Thus, by Theorem 1.2, \mathcal{FL}_R has CSL.

Example 1.10. Let G be a finite group and F a field.

- (i) If the characteristic of F does not divide the order of G , then FG is semisimple and hence the Gabriel quiver is totally disconnected (no arrows).
- (ii) If the characteristic of F is a prime number p and G is a finite p -group, then FG is a local ring and again, the Gabriel quiver is totally disconnected (the only arrow is a loop).
- (iii) In the case where the number of simples is different from the number of blocks, there is a block where two nonisomorphic simples extend, and we get a proper arrow in the Gabriel quiver.

Thus, in cases (i) and (ii), but not (iii), \mathcal{FL}_{FG} has CSL.

2. CSL FOR FINITE LENGTH MODULES

In this section we give a proof of Theorem 1.2. First, assume that \mathcal{FL}_R has CSL. As a consequence of the next lemma, the right Gabriel quiver of R must be totally disconnected.

Lemma 2.1. *Suppose $0 \rightarrow T \rightarrow M \rightarrow S \rightarrow 0$ is a non-split short exact sequence in Mod_R where S and T are nonisomorphic simple modules. Then $\text{End}(M_R)$ is isomorphic to a division subring of $\text{End}(S_R)$ and of $\text{End}(T_R)$.*

Proof. Since $\text{Hom}_R(M, T) = \{0\}$, $\text{Hom}_R(M, M)$ embeds in $\text{Hom}_R(M, S)$; since $\text{Hom}_R(T, S) = \{0\}$, we can identify $\text{Hom}_R(M, S)$ with $\text{Hom}_R(S, S)$. This yields a ring monomorphism $\text{End}(M_R) \rightarrow \text{End}(S_R)$. Similarly, since $\text{Hom}_R(S, M) = \{0\}$ and $\text{Hom}_R(T, S) = \{0\}$, we obtain a ring monomorphism $\text{End}(M_R) \rightarrow \text{End}(T_R)$. Thus $\text{End}(M_R)$ is isomorphic to a subring of the division rings $\text{End}(S_R)$ and $\text{End}(T_R)$, so $\text{End}(M_R)$ is a domain. Being also semiprimary, $\text{End}(M_R)$ is a division ring. \square

For the converse, we assume that the right Gabriel quiver of R is totally disconnected. We first show that every finite length indecomposable module is isotypic, and then that every isotypic module is either simple or admits a nonzero nilpotent endomorphism. Note that some of the results apply both to finite length modules over an arbitrary ring and to arbitrary modules over a semiprimary ring. We will use these results again in the next section.

Definition 2.2. A module M is *isotypic* if all simple subquotients of M are isomorphic. A sequence

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

of submodules is called an *isotypic filtration* of M of length ℓ if for every $i = 1, \dots, \ell$, the quotient M_i/M_{i-1} is isotypic.

Proposition 2.3. *Suppose R is a ring whose right Gabriel quiver is totally disconnected. Suppose that either*

- (i) M is an object of \mathcal{FL}_R , or
- (ii) R is semiprimary, and M is an object of Mod_R .

Then M is a finite direct sum of isotypic modules.

Proof. Step 1: The module M has an isotypic filtration. For example, the radical filtration of M can be refined to an isotypic filtration.

Step 2: For each isotypic filtration

$$0 \subset M_1 \subset \cdots \subset M_\ell = M$$

there is an isotypic filtration $0 \subset M'_1 \subset \cdots \subset M'_\ell = M$ such that $M_i \subseteq M'_i$ for each i and $\text{Hom}_R(M'_1, M/M'_1) = \{0\}$. Zorn's Lemma can be applied to the set of all isotypic submodules of M that contain M_1 ; let M'_1 be a maximal member of this set. We have $\text{Hom}_R(M'_1, M/M'_1) = \{0\}$ since the socle of M/M'_1 cannot contain a simple summand isomorphic to a subquotient of M'_1 . For $i > 1$, put $M'_i = M_i + M'_1$. Then

$$\frac{M'_i}{M'_{i-1}} = \frac{M_i + M'_1}{M_{i-1} + M'_1} \cong \frac{M_i}{M_i \cap (M_{i-1} + M'_1)} = \frac{M_i}{M_{i-1} + (M_i \cap M'_1)}$$

is epimorphic image of M_i/M_{i-1} and hence isotypic.

Step 3: Let M and N be isotypic modules that both satisfy (i) or (ii) of the proposition and for which $\text{Hom}_R(M, N) = \{0\}$. Then $\text{Ext}_R^1(M, N) = \{0\}$. When M and N are semisimple, this follows from the hypothesis on the right Gabriel quiver. The general case follows by induction on the lengths of semisimple filtrations of M and N .

Step 4: The module M is a direct sum of isotypic modules. We induct on the length ℓ of the isotypic filtration of M produced in Step 2. The case $\ell = 1$ is trivial. For the induction step, let M have an isotypic filtration $0 \subset M'_1 \subset \cdots \subset M'_{\ell+1} = M$. By inductive hypothesis, $M/M'_1 \cong \bigoplus_i M''_i$ is a direct sum of isotypic modules M''_i . Since $\text{Hom}_R(M'_1, M''_i) = \{0\}$ for all i , by Step 3 we have

$$\text{Ext}_R^1(M'_1, M/M'_1) = \bigoplus_i \text{Ext}_R^1(M'_1, M''_i) = \{0\}$$

Thus, the short exact sequence $0 \rightarrow M'_1 \rightarrow M \rightarrow M/M'_1 \rightarrow 0$ splits, and $M \cong M'_1 \oplus \bigoplus_i M''_i$ is a direct sum of isotypic modules. \square

Next we prove a criterion for isotypic modules to admit a nonzero nilpotent endomorphism. The argument is adapted from the proof of [1, Theorem 1.2].

Lemma 2.4. *Let R be any ring, and let M be a right R -module. Then M has no nonzero semisimple direct summand if and only if $\text{soc}(M) \subseteq \text{rad}(M)$.*

Proof. If M has no nonzero semisimple direct summand, then every simple submodule is superfluous, whence $\text{soc}(M) \subseteq \text{rad}(M)$. Conversely, if M does have a nonzero semisimple direct summand, then M has a simple direct summand, which is contained in $\text{soc}(M)$ but not $\text{rad}(M)$, so $\text{soc}(M) \not\subseteq \text{rad}(M)$. \square

Proposition 2.5. *Suppose that M_R is a nonzero isotypic module that is not simple. Assume in addition that either M has finite length or R is a perfect ring. Then M has a nonzero nilpotent endomorphism.*

Proof. If M has a nonzero simple direct summand, the conclusion is clear; so assume otherwise. By Lemma 2.4, $\text{soc}(M) \subseteq \text{rad}(M)$. Now, M is nonzero and isotypic, and the hypotheses imply that $M/\text{rad}(M)$ is semisimple; therefore, there exists a nonzero homomorphism $f_0: M/\text{rad}(M) \rightarrow \text{soc}(M)$. The composite map

$$f: M \xrightarrow{\pi} M/\text{rad}(M) \xrightarrow{f_0} \text{soc}(M) \xrightarrow{\iota} M$$

(where π is the canonical epimorphism and ι the inclusion map) is a nonzero endomorphism of M satisfying $f^2 = 0$. \square

Theorem 1.2 is now established. The “only if” part follows from Lemma 2.1. The “if” part follows from Propositions 2.3 and 2.5.

3. CSL FOR ALL MODULES

To prove Theorem 1.6 we will show

$$(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

By Theorem 1.2, statements (i) and (ii) are equivalent for any ring R .

(ii) \Rightarrow (iii): According to Proposition 2.3, the module $R_R = \bigoplus_i P_i$ is a finite direct sum of indecomposable isotypic submodules P_i . Two such submodules are either isomorphic or have no nonzero homomorphisms between them. Thus, $R =$

$\text{End}(R_R)$ is a finite direct product of matrix rings over the local endomorphism rings of the P_i 's.

(iii) \Rightarrow (iv): Apply “(iv) \Rightarrow (iii)” of [1, Theorem 1.2]. \square

We now prove Proposition 1.5. Let A be an elementary algebra over a finite field k of p^n elements. Let K be an algebraically closed field of characteristic p and $\varphi: K \rightarrow K$ the Frobenius automorphism, given by $\alpha \mapsto \alpha^{p^n}$; we identify k with the fixed field of φ . Let $\Sigma = K[x, x^{-1}; \varphi]$ be the V-ring studied in [4]; we claim that the ring $R = \Sigma \otimes_k A$ has the required properties.

(i) We infer from [13, Theorem A] that the categories of \mathcal{FL}_A and \mathcal{FL}_R are equivalent.

(ii) Follows from (i) and (iii).

(iii) Applying Theorem 1.6 to the semiprimary ring A , we deduce that \mathcal{Mod}_A has CSL. To see that R does not have CSL, note that since A is elementary, the ring homomorphism $\pi: A \rightarrow k$ gives rise to a surjective ring homomorphism

$$\pi \otimes 1: R = A \otimes_k \Sigma \longrightarrow k \otimes_k \Sigma.$$

Since \mathcal{Mod}_Σ does not have CSL (as explained in Example 1.4), and Σ is isomorphic to a factor ring of R , \mathcal{Mod}_R does not have CSL. \square

4. SOME QUESTIONS

The rings in Examples 1.4 and 1.8 are both simple noetherian domains. In light of the diametrically different behavior in the two examples, we pose the following question.

Question 4.1. *For which simple noetherian domains R does \mathcal{FL}_R have CSL?*

Question 4.2. *When do other subcategories of \mathcal{Mod}_R have CSL? What are the conditions under which all artinian modules have CSL? All noetherian modules? Is there an example of a ring which has CSL for finite length modules, but not CSL for artinian modules?*

One may also consider categories with quasi-CSL in the following sense:

Definition 4.3. Let \mathcal{C} be a category of modules, i.e. \mathcal{C} is a full subcategory of \mathcal{Mod}_R for some ring R . We say that an object M of \mathcal{C} is *quasi-simple* if the only submodules $N \subseteq M$ such that N and M/N are objects of \mathcal{C} are $N = 0$ and $N = M$. The category \mathcal{C} is said to have *quasi-CSL* if the only modules with endomorphism ring a division ring are the quasi-simple ones.

Question 4.4. *Are there interesting categories with quasi-CSL?*

Example 4.5. For a given ring R , the category \mathcal{FL}_R or the category \mathcal{Mod}_R has quasi-CSL if and only if it has CSL.

Nevertheless, in general *quasi-CSL* and *CSL* are different conditions on a module category, as will be seen in Example 4.8 below. We preface this example with some motivating observations.

Example 4.6. Even if R has CSL, then the ring $U_2(R)$ of upper triangular 2 by 2 matrices with coefficients in R need not have CSL. Indeed, if S is a simple right

R -module, then the row $(S \ S)$ is a right module over $U_2(R)$ of length 2 with endomorphism ring $\text{End}(S_R)$.

In fact, the category $\mathcal{M}od_{U_2(R)}$ is just the category of all maps between right R -modules, a map $f: A \rightarrow B$ being a module over $U_2(R)$ via

$$(a, b) \cdot \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = (ax, f(a)y + bz).$$

Consider the full subcategory $\mathcal{S}(R)$ of $\mathcal{M}od_{U_2(R)}$ consisting of all maps which are monomorphisms (“ \mathcal{S} =submodules”).

Question 4.7. *If R has CSL, does $\mathcal{S}(R)$ have quasi-CSL?*

Categories of type $\mathcal{S}(R)$ play a role in applications of ring theory; for example, the embeddings of a subgroup in a finite abelian group, or the embeddings of a subspace in a vector spaces such that the subspace is invariant under the action of a linear operator, fall into this type of category.

Example 4.8. For Λ a commutative uniserial ring with radical generator p and radical factor field k , the category $\mathcal{S}(\Lambda)$ has quasi-CSL but not CSL, as follows. There are exactly two quasi-simple modules, $S_1 = (k \ k)$ and $S_2 = (0 \ k)$, up to isomorphy; both have endomorphism ring k . Then $\mathcal{S}(\Lambda)$ has quasi-CSL since any embedding $(A \ B)$ with B a semisimple Λ -module is a direct sum of copies of S_1 and S_2 . On the other hand, if B is not semisimple then multiplication by p is a nonzero nilpotent endomorphism.

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